

# On Existence of Truthful Fair Cake Cutting Mechanisms

**Biaoshuai Tao**

John Hopcroft Center for Computer Science  
Shanghai Jiao Tong University

## Abstract

We study the fair division problem on divisible heterogeneous resources (the cake cutting problem) with strategic agents, where each agent can manipulate his/her private valuation in order to receive a better allocation. A (direct-revelation) mechanism takes agents' reported valuations as input, and outputs an allocation that satisfies a given fairness requirement. A natural and fundamental open problem, first raised by Chen et al. [12] and subsequently raised by Procaccia [21], Aziz and Ye [4], Brânzei and Miltersen [11], Menon and Larson [19], Bei et al. [6, 7], etc., is whether there exists a deterministic, truthful and envy-free (or even proportional) cake cutting mechanism. In this paper, we resolve this open problem by proving that there does not exist a deterministic, truthful and proportional cake cutting mechanism, even in the special case where all of the followings hold:

- there are only two agents;
- each agent's valuation is a piecewise-constant function;
- each agent is hungry: each agent has a strictly positive value on any part of the cake.

The impossibility result extends to the case where the mechanism is allowed to leave some part of the cake unallocated.

To circumvent this impossibility result, we aim to design mechanisms that possess certain degree of truthfulness. Motivated by the kind of truthfulness possessed by the classical I-cut-you-choose protocol, we define a weaker notion of truthfulness: *the risk-averse truthfulness*. We show that the well-known moving-knife procedure and Even-Paz algorithm do not have this truthful property. We propose a mechanism that is risk-averse truthful and envy-free, and a mechanism that is risk-averse truthful and proportional that always outputs allocations with connected pieces.

## 1 Introduction

The cake cutting problem studies the allocation of a piece of divisible heterogeneous resource to multiple agents, normally with a given fairness requirement. The cake is a metaphor for divisible heterogeneous resources, which is normally modeled as an interval  $[0, 1]$ . Different agents have different valuations on different parts of the interval. Typically, each agent's valuation is described by a *value density function*  $f : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$ , and his/her value on a subset  $X \subseteq [0, 1]$  is given by the Riemann integral  $\int_X f(x)dx$ . Starting with Thomson [24], the cake cutting problem has been widely studied by mathematicians, economists and computer scientists. See the books Brams et al. [9], Robertson and Webb [22] and Part II of Brandt et al. [10] and the survey Procaccia [21].

Two of the most widely studied fairness criteria are *proportionality* and *envy-freeness*. An allocation is proportional if each agent believes (s)he receives a share with a value that is at least  $\frac{1}{n}$  fraction of the value of the entire cake (where  $n$  is the number of the agents). An allocation is envy-free if each agent believes (s)he receives a share that has weakly more value than the share allocated to each of the other agents (i.e., an agent does not envy any other agents). Formal definitions for the two notions are in Sect. 2. If we require that the entire cake needs to be allocated (i.e., discarding some part of the cake is disallowed), an envy-free allocation is always proportional. It is well-known that envy-free allocations always exist [8], even if we require each agent must receive a connected interval [14]. In addition to the existences, the algorithm design aspect has also been considered in a long history [13, 15, 23, 2, 3]. In particular, we know how to compute a proportional allocation [13, 15] and an envy-free allocation [3] for any number of agents.

However, a fundamental issue when deploying a certain cake cutting algorithm is that agents are self-interested and may manipulate and misreport their valuations to the algorithm in order to get a better allocation. This motivates the study of the cake-cutting problem from a game theory aspect, in particular, a mechanism design aspect. Is there a *truthful* and fair cake cutting mechanism such that truth-telling is each agent's dominant strategy? This question was first proposed by Chen et al. [12].

To answer this question, we first need to address the following issue: how can we represent a value density function succinctly? Two different approaches have been considered in the past literature. In the first approach (e.g., [8, 22, 23, 16, 2, 3]), the mechanism communicates with the agents by a query model called *the Robertson-Webb query model*, where the mechanism learns the valuation of each agent through a sequence of queries that are of the following two types:

- **Eval** $_i(x, y)$ : ask agent  $i$  his/her value on the interval  $[x, y]$ ;
- **Cut** $_i(x, r)$ : ask agent  $i$  for a point  $y$  where  $[x, y]$  is worth exactly  $r$ .

In the second approach (e.g., [20, 12, 5, 19, 6, 7]), the value density function is assumed to be *piecewise-constant*. Piecewise-constant functions can approximate most natural real functions arbitrarily closely, and it can be succinctly encoded. The mechanism then takes the  $n$  encoded value density functions as input, and outputs an allocation. These mechanisms are called *direct revelation* mechanisms.

In the setting with the Robertson-Webb query model, the game agents are playing is an *extensive-form game*, whereas, in the piecewise-constant valuation setting, this is a one-round game where all the agents report their valuations simultaneously. Naturally, when truthfulness is concerned, agents in the first setting have much more room for manipulation. Indeed, for the first setting, Kurokawa et al. [16] proves that there is no truthful and envy-free mechanism that requires a bounded number of Robertson-Webb queries. A strong impossibility result by Brânzei and Miltersen [11] shows that, for any truthful mechanism, there exists an agent who receives a zero value. In particular, when there are only two agents, the only truthful mechanism is essentially the one that allocates the entire cake to a single agent.

For direct revelation mechanisms, Chen et al. [12] gives the first truthful envy-free cake cutting mechanism that works when each agent's valuation is *piecewise-uniform*, a special case of piecewise-constant valuations with the additional assumption that each value density function takes value either 0 or 1. Chen et al. [12] then proposes the following natural open problem.

**Problem 1.** Does there exist a (deterministic) truthful, envy-free (or even proportional) cake cutting mechanism for piecewise-constant value density functions?

There are many partial progresses on this problem. Aziz and Ye [4] shows that there exists no truthful mechanism that satisfies either one of the following properties:

- Proportional and Pareto-optimal;
- Robust-proportional and non-wasteful (non-wasteful means that no piece is allocated to an agent who does not want it, a notion weaker than Pareto-optimality).

Menon and Larson [19] shows that there exists no truthful mechanism that is even approximately-proportional, with the constraint that each agent must receive a connected piece. Bei et al. [6] shows that there exists no truthful, proportional mechanism under any one of the following three settings:

- the mechanism is non-wasteful;
- the mechanism is position-oblivious (meaning that the allocation of a cake-part is based only on the agents' valuations of that part, and not on its relative position on the cake);
- agents report the value density functions sequentially, where an agent's strategy can depend on the reports of the previous agents.

On the positive side, the mechanism proposed by Chen et al. [12] for piecewise-uniform value density functions is further studied by Maya and Nisan [18] and Li et al. [17]. Maya and Nisan [18] characterizes truthful mechanisms and shows that the mechanism proposed by Chen et al. [12] is unique in some sense. Li

et al. [17] shows that this mechanism also works in the setting where agents have externalities. Alijani et al. [1] proposes some mechanisms that work for special cases of piecewise-uniform functions with small number of cuts. Bei et al. [7] proposes a truthful envy-free mechanism for piecewise-uniform value density functions that do not need the *free-disposal* assumption, an assumption made in the mechanism by Chen et al. [12]. As can be seen above, most positive results are regarding piecewise-uniform valuations.

All the mechanisms mentioned above are deterministic. If we allow randomized mechanisms, a simple mechanism proposed by Mossel and Tamuz [20] is universal envy-free and truthful in expectation. However, randomized mechanisms have many drawbacks. Firstly, agents can be risk-seeking or risk-averse, and may have different views on a truthful-in-expectation randomized mechanisms. Secondly, agents may have concern on the source of the randomness. It is costly to find a trustworthy random source. Agents receiving less utility due to randomness may believe they have not been treated fairly.

Despite those above-mentioned progresses, Problem 1 remains open.

## 1.1 Our Results

As the main result of this paper, we resolve Problem 1 by proving that there does not exist a (deterministic) truthful proportional cake cutting mechanism. This impossibility result can be extended to the setting where there are only two agents, each agent has a strictly positive value on any part of the cake (we say that the agents are *hungry* in this case), and the mechanism is allowed to leave some part of the cake unallocated.

To circumvent this impossibility result, we propose a weaker truthful notion called *risk-averse truthful*. This is motivated by the truthful guarantee of the *I-cut-you-choose* protocol: the protocol works for two agents; the first agent cuts the cake into two pieces (s)he believes to have equal value; the second agent selects the piece with a higher value, and the remaining piece is given to the first agent. The truthfulness of the second agent is apparent. For the first agent, (s)he has no incentive to manipulate without the knowledge of the second agent's valuation, for otherwise there is a risk for him/her to receive a piece less than a half of the entire cake. Our risk-averse truthful notion captures the risk-averseness of the agents and the setting where an agent does not know other agents' valuations.

We show that those well-known algorithms, e.g., the moving-knife procedure [13] and the Even-Paz algorithm [15], do not satisfy this truthful property. We then propose a mechanism that is risk-averse truthful and envy-free, and a mechanism that is risk-averse truthful and proportional that always outputs allocations with connected pieces.

## 2 Preliminary

The cake is modeled as the interval  $[0, 1]$ , which is allocated to  $n$  agents. Each agent  $i$  has a *value density function*  $f_i : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$  that describes his/her preference on the cake. A value density function  $f_i$  is *piecewise-constant* if  $[0, 1]$  can be partitioned into finitely many intervals, and  $f_i$  is constant on each of these intervals. We will assume agents' value density functions are piecewise-constant throughout the paper, although our result in Sect. 6 do not rely on this. Agent  $i$  is *hungry* if  $f_i(x) > 0$  for any  $x \in [0, 1]$ . Given a subset  $X \subseteq [0, 1]$ , agent  $i$ 's *utility* on  $X$ , denoted by  $v_i(X)$ , is given by

$$v_i(X) = \int_X f_i(x) dx.$$

An allocation  $(A_1, \dots, A_n)$  is a collection of mutually disjoint subsets of  $[0, 1]$ , where  $A_i$  is the subset allocated to agent  $i$ . An allocation is *entire* if  $\bigcup_{i=1}^n A_i = [0, 1]$ . Notice that an impossibility result without the entire requirement is stronger than an impossibility result with this requirement. An allocation is *proportional* if each agent receives his/her average share of the entire cake:

$$\forall i : v_i(A_i) \geq \frac{1}{n} v_i([0, 1]).$$

An allocation is *envy-free* if each agent receive a portion that has weakly higher value than any portion received by any other agent, based on his/her own valuation:

$$\forall i, j : v_i(A_i) \geq v_i(A_j).$$

An entire envy-free allocation is always proportional. In the case with two agents, if an allocation is entire, it is envy-free if and only if it is proportional. In Sect. 6, we consider a specific kind of allocations where each agent needs to receive a connected piece of cake, i.e., each  $A_i$  is an *interval*.

A *mechanism* is a function  $\mathcal{M}$  that maps  $n$  value density functions  $F = (f_1, \dots, f_n)$  to an allocation  $(A_1, \dots, A_n)$ . Given  $\mathcal{M}(F) = (A_1, \dots, A_n)$ , we write  $\mathcal{M}_i(F) = A_i$ . That is,  $\mathcal{M}_i(F)$  outputs the share allocated to agent  $i$ , given input  $F = (f_1, \dots, f_n)$ . A mechanism is proportional/envy-free if it always outputs a proportional/envy-free allocation with respect to the input  $F = (f_1, \dots, f_n)$ . A mechanism is entire if it always outputs entire allocations. In this paper, we consider only deterministic mechanisms.

A mechanism  $\mathcal{M}$  is *truthful* if each agent's dominant strategy is to report his/her true value density function. That is, for each  $i \in [n]$ , any  $(f_1, \dots, f_n)$  and any  $f'_i$ ,

$$v_i(\mathcal{M}_i(f_1, \dots, f_n)) \geq v_i(\mathcal{M}_i(f_1, \dots, f_{i-1}, f'_i, f_{i+1}, \dots, f_n)).$$

As a clarification, when proportionality/envy-freeness is concerned, a mechanism must output an allocation that is proportional/envy-free with respect to the *reported value density functions*; when truthfulness is concerned, we require each agent's misreporting does not give this agent strictly more utility, and the utility here is with respect to this agent's *true value density function*.

### 3 Impossibility Result for Truthful Proportional Mechanism

In this section, we prove the following theorem.

**Theorem 3.1.** *There does not exist a truthful proportional mechanism, even when all of the followings hold:*

- *there are two agents;*
- *each agent's value density function is piecewise-constant;*
- *each agent is hungry: each  $f_i$  satisfies  $f_i(x) > 0$  for any  $x \in [0, 1]$ ;*
- *the mechanism needs not to be entire.*

We will prove Theorem 3.1 by contradiction. Suppose there exists a truthful proportional mechanism  $\mathcal{M}$  for two agents. For a description of the main idea behind the proof, we construct multiple cake cutting instances, analyze the outputs of  $\mathcal{M}$  on these instances, and prove that truthfulness and proportionality cannot be both guaranteed on all these instances. In particular, we will construct six instances. For the first five instances, we show that the outputs of  $\mathcal{M}$  are unique. Based on the outputs for the first five instances, we show that any allocation output by  $\mathcal{M}$  for the sixth instance will violate either proportionality or truthfulness. The six instances constructed are shown in Table 1.

We start with the simplest cake cutting instance.

**Instance 1.**  $F^1 = (f_1^1, f_2^1)$ , where  $f_1^1(x) = 1$  and  $f_2^1(x) = 1$  for  $x \in [0, 1]$ .

To ensure proportionality, we must have  $|\mathcal{M}_1(F^1)| = |\mathcal{M}_2(F^1)| = \frac{1}{2}$ . We will denote the allocation of  $\mathcal{M}_1(F^1)$  by  $(X_1, X_2)$ .  $X_1$  and  $X_2$  will be used multiple times in the definitions of other instances. It is helpful to assume  $X_1 = [0, 0.5]$  and  $X_2 = (0.5, 1]$ .

**Definition 3.2.**  $X_1 = |\mathcal{M}_1(F^1)|$  and  $X_2 = |\mathcal{M}_2(F^1)|$ .

In the instances constructed later, we let  $\varepsilon > 0$  be a sufficiently small real number.

Next, we consider the following instance.

**Instance 2.**  $F^2 = (f_1^2, f_2^2)$ , where  $f_1^2(x) = 1$  for  $x \in [0, 1]$  and

$$f_2^2(x) = \begin{cases} \varepsilon & x \in X_1 \\ 1 & x \in X_2 \end{cases}.$$

The following proposition shows that the only possible allocation output by  $\mathcal{M}$  for Instance 2 is  $(X_1, X_2)$ .

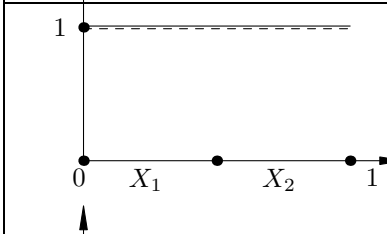
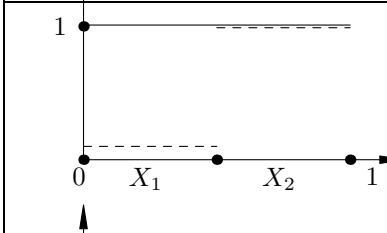
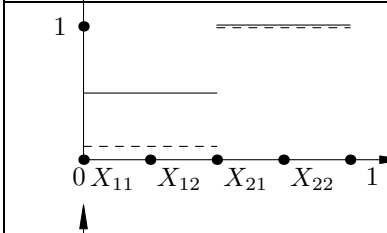
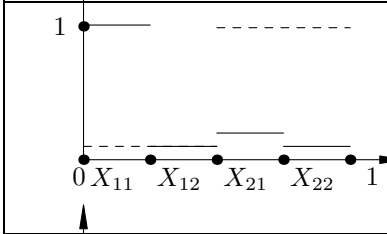
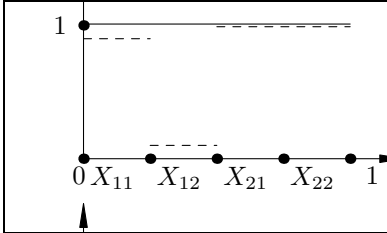
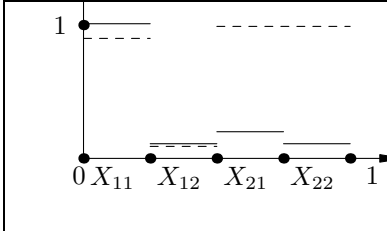
Instance	Allocation
	$\mathcal{M}(F^1) = (X_1, X_2)$
	$\mathcal{M}(F^2) = (X_1, X_2)$
	$\mathcal{M}(F^3) = (X_{11} \cup X_{21}, X_{12} \cup X_{22})$
	$\mathcal{M}(F^4) = (X_{11} \cup X_{21}, X_{12} \cup X_{22})$
	$\mathcal{M}(F^5) = (X_1, X_2)$
	See Sect. 3.1

Table 1: Instances constructed for the proof of Theorem 3.1 and the corresponding allocations given by  $\mathcal{M}$ . The value density for agent 1 is shown in solid lines, and the value density for agent 2 is shown in dashed lines.

**Proposition 3.3.**  $\mathcal{M}(F^2) = (X_1, X_2)$ .

*Proof.* Firstly, we must have  $|\mathcal{M}_2(F^2)| \leq \frac{1}{2}$ . Otherwise, agent 1 will receive a subset of length strictly less than  $1/2$ . Since agent 1's valuation is uniform on  $[0, 1]$ ,  $\mathcal{M}$  is not proportional.

Secondly, we must have  $X_2 \subseteq \mathcal{M}_2(F^2)$ . Suppose agent 2 does not receive all of  $X_2$ , i.e.,  $|X_2 \cap \mathcal{M}_2(F^2)| < \frac{1}{2}$ . Given that  $|\mathcal{M}_2(F^2)| \leq \frac{1}{2}$ , we have

$$v_2(\mathcal{M}_2(F^2)) = v_2(X_1 \cap \mathcal{M}_2(F^2)) + v_2(X_2 \cap \mathcal{M}_2(F^2)) \leq \varepsilon \cdot \left(\frac{1}{2} - |X_2 \cap \mathcal{M}_2(F^2)|\right) + 1 \cdot |X_2 \cap \mathcal{M}_2(F^2)| < \frac{1}{2}.$$

On the other hand, if agent 2 misreports his/her value density function to  $f_2^1$  (instead of his/her true value density function  $f_2^2$ ), the mechanism receives input  $(f_1^2, f_2^1)$ , which becomes Instance 1 since  $f_1^1 = f_1^2$ . In this case the allocation output is  $(X_1, X_2)$ , and agent 2's total value, in terms of his true valuation  $f_2^2$ , is  $\frac{1}{2}$ . Therefore, agent 2 can receive more value by misreporting his/her value density function, and  $\mathcal{M}$  cannot be truthful.

Putting together, we have  $X_2 \subseteq \mathcal{M}_2(F^2)$  and  $|\mathcal{M}_2(F^2)| \leq \frac{1}{2}$ , which implies  $\mathcal{M}_2(F^2) = X_2$ . Agent 1 will then receive the remaining part of the cake which is just enough to guarantee proportionality:  $\mathcal{M}_1(F^2) = X_1$ .  $\square$

The next instance we consider is slightly more complicated.

**Instance 3.**  $F^3 = (f_1^3, f_2^3)$ , where

$$f_1^3(x) = \begin{cases} 0.5 & x \in X_1 \\ 1 & x \in X_2 \end{cases} \quad \text{and} \quad f_2^3(x) = \begin{cases} \varepsilon & x \in X_1 \\ 1 & x \in X_2 \end{cases}.$$

The following proposition shows that agent 1's allocation is exactly the union of half of  $X_1$  and half of  $X_2$ .

**Proposition 3.4.**  $|\mathcal{M}_1(F^3) \cap X_1| = |\mathcal{M}_1(F^3) \cap X_2| = |\mathcal{M}_2(F^3) \cap X_1| = |\mathcal{M}_2(F^3) \cap X_2| = \frac{1}{4}$ .

We provide a brief intuition behind the proof first. Firstly, agent 1 cannot receive a subset of length more than 0.5. Otherwise, in Instance 2, agent 1 will misreport his value density function from  $f_1^2$  to  $f_1^3$ , which is more beneficial to agent 1 (as  $f_1^2$  is uniform and agent 1 receives a larger length by misreporting).

Secondly, agent 1 cannot receive less than half of  $X_2$ . If agent 1 receives less than half of  $X_2$  by a length of  $x$ , agent 1 needs to receive more than half of  $X_1$  by a length of at least  $2x$  to guarantee proportionality. This will make the total length received by agent 1 more than 0.5.

Thirdly, agent 1 cannot receive more than half of  $X_2$ . If agent 1 receives more than half of  $X_2$ , agent 2, having significantly less value on  $X_1$ , will have to receive a length on  $X_1$  that is significantly longer than half of  $X_1$ . This will destroy the proportionality of agent 1 for that agent 2 has already taken too much.

Finally, having shown that agent 1 must receive exactly half of  $X_2$ , the proportionality of agent 1 and the proven fact that agent 1's received total length is at most 0.5 imply that agent 1 has to receive exactly half of  $X_1$ .

*Proof of Proposition 3.4.* Firstly, we must have  $|\mathcal{M}_1(F^3)| \leq \frac{1}{2}$ . Suppose this is not the case:  $|\mathcal{M}_1(F^3)| > \frac{1}{2}$ . We show that  $\mathcal{M}$  cannot be truthful. Consider Instance 2 where agent 1's value density function is uniform. In Instance 2, if agent 1 misreports his/her value density function to  $f_1^3$ , the mechanism  $\mathcal{M}$  will see an input that is exactly the same as  $F^3$  (notice  $f_2^2 = f_2^3$ ), and agent 1 will receive a subset with length strictly more than  $\frac{1}{2}$ . However, we have seen in Proposition 3.3 that agent 1 will receive a subset with length exactly  $\frac{1}{2}$  if (s)he reports truthfully. Since agent 1's true valuation is uniform, agent 1 will benefit from this misreporting.

Let  $|\mathcal{M}_1(F^3) \cap X_2| = \frac{1}{4} + x$  where  $x \in [-\frac{1}{4}, \frac{1}{4}]$ . We aim to show that  $x = 0$ . Agent 1's total utility on  $[0, 1]$  is  $\int_0^1 f_1^3(x) dx = \frac{3}{4}$ . To guarantee proportionality, we must have

$$v_1(\mathcal{M}_1(F^3)) = v_1(\mathcal{M}_1(F^3) \cap X_1) + v_1(\mathcal{M}_1(F^3) \cap X_2) = 0.5 \cdot |\mathcal{M}_1(F^3) \cap X_1| + 1 \cdot \left(\frac{1}{4} + x\right) \geq \frac{3}{8}. \quad (1)$$

By rearranging (1), we have  $|\mathcal{M}_1(F^3) \cap X_1| \geq \frac{1}{4} - 2x$ . The total length agent 1 receives is then  $|\mathcal{M}_1(F^3)| = |\mathcal{M}_1(F^3) \cap X_1| + |\mathcal{M}_1(F^3) \cap X_2| \geq \frac{1}{2} - x$ . Since we have seen  $|\mathcal{M}_1(F^3)| \leq \frac{1}{2}$  at the beginning, we have  $x \geq 0$ .

On the other hand, since  $|\mathcal{M}_1(F^3) \cap X_2| = \frac{1}{4} + x$ , we have  $|\mathcal{M}_2(F^3) \cap X_2| \leq \frac{1}{4} - x$ . Since  $v_2([0, 1]) = \frac{1}{2} + \frac{1}{2}\varepsilon$  and  $v_2(\mathcal{M}_2(F^3) \cap X_2) = 1 \cdot |\mathcal{M}_2(F^3) \cap X_2| \leq \frac{1}{4} - x$ , to guarantee proportionality for agent 2, we must have  $v_2(\mathcal{M}_2(F^3) \cap X_1) \geq \frac{1}{4}\varepsilon + x$ . Therefore,  $|\mathcal{M}_2(F^3) \cap X_1| \geq \frac{1}{4} + \frac{x}{\varepsilon}$ , which implies  $|\mathcal{M}_1(F^3) \cap X_1| \leq \frac{1}{4} - \frac{x}{\varepsilon}$ . Substituting this into (1), we have

$$0.5 \cdot \left( \frac{1}{4} - \frac{x}{\varepsilon} \right) + \left( \frac{1}{4} + x \right) \geq \frac{3}{8},$$

which implies  $x \leq 0$  if  $\varepsilon$  is sufficiently small.

Therefore,  $x = 0$ , and we have  $|\mathcal{M}_1(F^3) \cap X_2| = \frac{1}{4}$ . Since agent 1 receives exactly length  $\frac{1}{4}$  on  $X_2$ , to guarantee proportionality, agent 1 must receive at least length  $\frac{1}{4}$  on  $X_1$ . To guarantee  $|\mathcal{M}_1(F^3)| \leq \frac{1}{2}$ , agent 1 must receive at most length  $\frac{1}{4}$  on  $X_1$ . Therefore, we have  $|\mathcal{M}_1(F^3) \cap X_1| = \frac{1}{4}$ .

Finally, agent 2 must receive the remaining part of the cake to guarantee proportionality.  $\square$

We will define four subsets  $X_{11}, X_{12}, X_{21}, X_{22}$  of  $[0, 1]$  that will be used for constructing other instances later.

**Definition 3.5.**  $X_{11} = |\mathcal{M}_1(F^3) \cap X_1|$ ,  $X_{12} = |\mathcal{M}_2(F^3) \cap X_1|$ ,  $X_{21} = |\mathcal{M}_1(F^3) \cap X_2|$  and  $X_{22} = |\mathcal{M}_2(F^3) \cap X_2|$ .

Proposition 3.4 implies  $|X_{11}| = |X_{12}| = |X_{21}| = |X_{22}| = \frac{1}{4}$ . It is helpful for the readers to assume  $X_{11} = [0, 0.25]$ ,  $X_{12} = (0.25, 0.5]$ ,  $X_{21} = (0.5, 0.75]$  and  $X_{22} = (0.75, 1]$ .

**Instance 4.**  $F^4 = (f_1^4, f_2^4)$ , where

$$f_1^4(x) = \begin{cases} 1 & x \in X_{11} \\ \varepsilon & x \in X_{12} \\ 2\varepsilon & x \in X_{21} \\ \varepsilon & x \in X_{22} \end{cases} \quad \text{and} \quad f_2^4(x) = \begin{cases} \varepsilon & x \in X_1 \\ 1 & x \in X_2 \end{cases}.$$

We will show that  $\mathcal{M}(F^3)$  and  $\mathcal{M}(F^4)$  output the same allocation.

**Proposition 3.6.**  $\mathcal{M}_1(F^4) = X_{11} \cup X_{21}$  and  $\mathcal{M}_2(F^4) = X_{12} \cup X_{22}$ .

*Proof.* Noticing that  $f_2^2 = f_2^3 = f_2^4$ , for the same reason in the proof of Proposition 3.4, we must have  $|\mathcal{M}_1(F^4)| \leq \frac{1}{2}$ . Otherwise, agent 1 in Instance 2 will misreport his/her true value density function  $f_1^2$  to  $f_1^4$ .

On the other hand, if agent 1 misreports his/her true value density function  $f_1^4$  to  $f_1^3$ , the mechanism  $\mathcal{M}$  will see the same input as  $F^3$  and allocate  $X_{11} \cup X_{21}$  to agent 1. With respect to agent 1's true valuation  $f_1^4$ , this is worth  $\frac{1}{4} + \frac{\varepsilon}{2}$ . To guarantee truthfulness, agent 1 must receive a value of at least  $\frac{1}{4} + \frac{\varepsilon}{2}$  on  $\mathcal{M}_1(F^4)$ :  $v_1(\mathcal{M}_1(F^4)) \geq \frac{1}{4} + \frac{\varepsilon}{2}$ .

Given that agent 1 can receive a subset of length at most  $\frac{1}{2}$ , the maximum value agent 1 can receive is  $\frac{1}{4} + \frac{\varepsilon}{2}$ , by receiving the two subsets  $X_{11}$  and  $X_{21}$  that are most valuable to agent 1. Therefore,  $|\mathcal{M}_1(F^4)| \leq \frac{1}{2}$  and  $v_1(\mathcal{M}_1(F^4)) \geq \frac{1}{4} + \frac{\varepsilon}{2}$  imply  $\mathcal{M}_1(F^4) = X_{11} \cup X_{21}$ .

Finally, to guarantee proportionality, agent 2 must receive the remaining part of the cake.  $\square$

**Instance 5.**  $F^5 = (f_1^5, f_2^5)$ , where  $f_1^5(x) = 1$  for  $x \in [0, 1]$  and

$$f_2^5(x) = \begin{cases} 1 - \varepsilon & x \in X_{11} \\ \varepsilon & x \in X_{12} \\ 1 & x \in X_2 \end{cases}.$$

We show that there is only one possible output for  $\mathcal{M}(F^5)$  that guarantee both truthfulness and proportionality, with  $\mathcal{M}(F^5) = \mathcal{M}(F^1) = \mathcal{M}(F^2)$ .

**Proposition 3.7.**  $\mathcal{M}_1(F^5) = X_1$  and  $\mathcal{M}_2(F^5) = X_2$ .

*Proof.* Firstly, we must have  $|\mathcal{M}_1(F^5)| \geq \frac{1}{2}$  to guarantee proportionality for agent 1. Therefore,  $|\mathcal{M}_2(F^5)| \leq \frac{1}{2}$ . Secondly, if agent 2 misreports his/her value density function to  $f_2^2$ , the mechanism  $\mathcal{M}$  will see an input exactly the same as  $F^2$ , and will allocate  $X_2$  to agent 2. This is worth  $\frac{1}{2}$  with respect to agent 2's true valuation  $f_2^5$ . Therefore, we must have  $v_2(\mathcal{M}_2(F^5)) \geq \frac{1}{2}$ , for otherwise agent 2 will misreport his/her value density function to  $f_2^2$ . Given that agent 2 can receive a length of at most  $\frac{1}{2}$ , the maximum value (s)he can receive is  $\frac{1}{2}$ , by receiving  $X_2$  that is most valuable to agent 2. Therefore,  $\mathcal{M}_2(F^5) = X_2$ . To guarantee proportionality for agent 1, we must also have  $\mathcal{M}_1(F^5) = X_1$ .  $\square$

Notice that, although we do not require entire allocations, the proportionality and truthfulness constraints make the output allocations of  $\mathcal{M}$  for the first five instances entire.

Finally, we will consider our last instance below, and show that  $\mathcal{M}$  cannot be both truthful and proportional for any allocation it outputs.

**Instance 6.**  $F^6 = (f_1^6, f_2^6)$ , where

$$f_1^6(x) = \begin{cases} 1 & x \in X_{11} \\ \varepsilon & x \in X_{12} \\ 2\varepsilon & x \in X_{21} \\ \varepsilon & x \in X_{22} \end{cases} \quad \text{and} \quad f_2^6(x) = \begin{cases} 1 - \varepsilon & x \in X_{11} \\ \varepsilon & x \in X_{12} \\ 1 & x \in X_2 \end{cases}.$$

We will analyze this instance in the following sub-section.

### 3.1 Analysis of $\mathcal{M}(F^6)$

We show that  $\mathcal{M}$  cannot output an allocation for Instance 6 that guarantees both truthfulness and proportionality. This will give us a contradiction, and proves Theorem 3.1. To show this, we begin by proving three propositions, and then show that they cannot be simultaneously satisfied.

**Proposition 3.8.**  $|\mathcal{M}_2(F^6) \cap X_2| \leq \frac{1}{4} + \frac{1}{4}\varepsilon$ .

*Proof.* Suppose this is not the case:  $|\mathcal{M}_2(F^6) \cap X_2| > \frac{1}{4} + \frac{1}{4}\varepsilon$ . Consider Instance 4. By Proposition 3.6, we have  $\mathcal{M}_2(F^4) = X_{12} \cup X_{22}$ , and agent 2 can receive value  $\frac{1}{4} + \frac{1}{4}\varepsilon$  (with respect to  $f_2^4$ ). By misreporting from  $f_2^4$  to  $f_2^6$ , the mechanism  $\mathcal{M}$  will see input  $F^6$  and allocate  $\mathcal{M}_2(F^6)$  to agent 2 with  $|\mathcal{M}_2(F^6) \cap X_2| > \frac{1}{4} + \frac{1}{4}\varepsilon$ . With respect to agent 2's true value density function  $f_2^4$  in Instance 4, this is worth more than  $\frac{1}{4} + \frac{1}{4}\varepsilon$ . Therefore,  $\mathcal{M}$  cannot be truthful.  $\square$

**Proposition 3.9.**  $v_1(\mathcal{M}_1(F^6)) \geq \frac{1}{4} + \frac{1}{4}\varepsilon$  with respect to  $f_1^6$ .

*Proof.* Suppose agent 1 misreports his/her true value density function  $f_1^6$  to  $f_1^5$ . The mechanism  $\mathcal{M}$  will see input  $F^5$ , which will allocate  $X_1$  to agent 1 by Proposition 3.7. This is worth  $\frac{1}{4} + \frac{1}{4}\varepsilon$  to agent 1. Therefore, to guarantee truthfulness, we must have  $v_1(\mathcal{M}_1(F^6)) \geq \frac{1}{4} + \frac{1}{4}\varepsilon$ .  $\square$

**Proposition 3.10.**  $v_2(\mathcal{M}_2(F^6)) \geq \frac{3}{8}$  with respect to  $f_2^6$ .

*Proof.* We have  $v_2([0, 1]) = \frac{1}{4}((1 - \varepsilon) + \varepsilon) + \frac{1}{2} \times 1 = \frac{3}{4}$ . The proposition follows by the proportionality of agent 2.  $\square$

We first give an intuitive argument to show that Proposition 3.8, 3.9 and 3.10 cannot be all satisfied. In  $F^6$ , agent 2 has a value equals to or approximately equals to 1 on each of the three segments  $X_{11}$ ,  $X_{21}$  and  $X_{22}$  and has a negligible value on  $X_{12}$ . Proposition 3.8 indicates that (s)he can receive at most (a little bit more than) half of  $X_{21} \cup X_{22}$ . To guarantee proportionality (indicated by Proposition 3.10), (s)he must receive approximately half of  $X_{11}$ . On the other hand, by our construction of  $f_1^6$ , it is easy to see that Proposition 3.9 indicates that almost entire  $X_{11}$  needs to be given to agent 1. This gives a contradiction.

Formally, Proposition 3.8 implies  $v_2(\mathcal{M}_2(F^6) \cap X_2) \leq \frac{1}{4} + \frac{1}{4}\varepsilon$ . Proposition 3.10 then indicates  $v_2(\mathcal{M}_2(F^6) \cap X_1) \geq \frac{1}{8} - \frac{1}{4}\varepsilon$ . Even if the entire  $X_{12}$  is allocated to agent 2 (which is worth  $\frac{1}{4}\varepsilon$ ), we still have

$$|\mathcal{M}_2(F^6) \cap X_{11}| \geq \frac{\frac{1}{8} - \frac{1}{4}\varepsilon - \frac{1}{4}\varepsilon}{1 - \varepsilon} = \frac{1 - 4\varepsilon}{8 - 8\varepsilon}.$$



For agent 1, we must then have

$$|\mathcal{M}_1(F^6) \cap X_{11}| \leq \frac{1}{4} - \frac{1-4\varepsilon}{8-8\varepsilon} = \frac{1+2\varepsilon}{8-8\varepsilon}.$$

To find an upper bound for  $v_1(\mathcal{M}_1(F^6))$ , suppose agent 1 receives all of  $X_{12}, X_{21}$  and  $X_{22}$ . Even in this case, we have the following upper bound for  $v_1(\mathcal{M}_1(F^6))$ :

$$v_1(\mathcal{M}_1(F^6)) \leq \frac{1+2\varepsilon}{8-8\varepsilon} \cdot 1 + \frac{1}{4} \cdot \varepsilon + \frac{1}{4} \cdot 2\varepsilon + \frac{1}{4} \cdot \varepsilon = \frac{1+2\varepsilon}{8-8\varepsilon} + \varepsilon.$$

Taking  $\varepsilon \rightarrow 0$ , the limit of the above upper bound is  $\frac{1}{8}$ . Thus,  $v_1(\mathcal{M}_1(F^6)) < \frac{1}{4} + \frac{1}{4}\varepsilon$  for sufficiently small  $\varepsilon$ , and Proposition 3.9 cannot be satisfied.

## 4 On Weaker Truthful Guarantees, Risk-Averse Truthfulness

We have seen in the previous section that standard dominant strategy truthfulness cannot be guaranteed if we want a proportional mechanism. Proportionality is one of the most basic criteria for fairness, and requiring each agent receiving at least an average share can be considered as a minimum fairness requirement in many applications. Therefore, we do not seek to relax this assumption in this paper. On the other hand, we will consider weaker truthful criteria.

A common truthful criteria is to require that the truth-telling profile form a *Nash Equilibrium*. In many applications, this is a significant weaker guarantee than dominant strategy truthfulness. However, in our cake cutting case, this truthful criteria is equivalent to the dominant strategy truthfulness, as the following theorem shows.

**Theorem 4.1.** *If a mechanism  $\mathcal{M}$  satisfies that agents' strategies of truthfully reporting their value density functions form a Nash equilibrium, then  $\mathcal{M}$  is (dominant strategy) truthful.*

*Proof.* Suppose  $\mathcal{M}$  satisfying this property is not dominant strategy truthful. Given a valuation profile  $(f_1, \dots, f_n)$ , there must exist an agent  $i$  and  $n-1$  value density functions  $f'_1, \dots, f'_{i-1}, f'_{i+1}, \dots, f'_n$  reported by the other  $n-1$  agents, such that reporting certain  $f'_i$  is more beneficial for agent  $i$  than truthfully reporting  $f_i$ . Now, consider a different valuation profile  $(f'_1, \dots, f'_{i-1}, f_i, f'_{i+1}, \dots, f'_n)$ . In this new profile, for each  $j \neq i$ , the function  $f'_j$ , being the reported function in the previous case, becomes the true valuation for agent  $j$ . In this new setting, if the remaining  $n-1$  agents truthfully report their value density functions, which are  $f'_1, \dots, f'_{i-1}, f'_{i+1}, \dots, f'_n$ , agent  $i$ 's best response is to report  $f'_i$  instead of his/her true valuation  $f_i$  (as we have seen in the first setting). This indicates that truth-telling is not a Nash equilibrium  $\square$

Even we do not have any progress on many standard truthful guarantees in game theory, there are still mechanisms that can achieve "certain degree of truthfulness" in practice. Most notably, the *I-cut-you-choose* protocol achieves some kind of truthfulness. The protocol works for proportional/envy-free cake cutting with two agents: agent 1 find a point  $x$  such that  $v_1([0, x]) = v_1([x, 1])$ ; agent 2 is allocated one of  $[0, x]$  and  $[x, 1]$  that is more valuable to him/her, and the other piece is allocated to agent 1. It is easy to see that agent 2's dominant strategy is truth-telling: (s)he has no control on the position of  $x$ , and truth-telling can ensure (s)he gets a piece with a larger value. On the other hand, although it is not a dominant strategy for agent 1 to tell the truth, agent 1 still does not have incentive to lie in the case (s)he has no knowledge on agent 2's valuation. If (s)he reports a value density function that results in a different position of  $x$ , there is always a risk that (s)he will receive a piece with a value less than 1/2 of the entire cake (i.e., less than the value guaranteed by proportionality).

There are two reasons behind agent 1's truth-telling incentive. Firstly, as mentioned, (s)he does not have prior knowledge on agent 2's valuations. Secondly, (s)he is a risk-averse agent: whenever there is a risk of receiving a value that does not meet the minimum proportional requirement, (s)he prefers to avoid the risk.

Motivated by this example, we define and consider a new truthful criterion: *the risk-averse truthfulness*.

**Definition 4.2.** A mechanism  $\mathcal{M}$  is *risk-averse truthful* if, for each agent  $i$  with value density function  $f_i$  and for any  $f'_i$ , either one of the following holds:

1. for any  $f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n$ ,

$$v_i(\mathcal{M}_i(f_1, \dots, f_{i-1}, f_i, f_{i+1}, \dots, f_n)) \geq v_i(\mathcal{M}_i(f_1, \dots, f_{i-1}, f'_i, f_{i+1}, \dots, f_n));$$

2. there exist  $f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n$  such that  $v_i(\mathcal{M}_i(f_1, \dots, f_{i-1}, f'_i, f_{i+1}, \dots, f_n)) < \frac{1}{n}v_i([0, 1])$ .

In other words, a mechanism is risk-averse truthful if either an agent's misreporting is non-beneficial, or the misreporting can potentially cause the agent receiving a piece with a value that is less than his/her proportional value.

We also define a weaker notion of risk-averse truthfulness, which, in the case where agent's misreporting may be beneficial, the misreporting can also make the agent receive less value than what (s)he will receive when (s)he is truth-telling.

**Definition 4.3.** A mechanism  $\mathcal{M}$  is *weakly risk-averse truthful* if, for each agent  $i$  with value density function  $f_i$  and for any  $f'_i$ , either one of the following holds:

1. for any  $f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n$ ,

$$v_i(\mathcal{M}_i(f_1, \dots, f_{i-1}, f_i, f_{i+1}, \dots, f_n)) \geq v_i(\mathcal{M}_i(f_1, \dots, f_{i-1}, f'_i, f_{i+1}, \dots, f_n));$$

2. there exist  $f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n$  such that

$$v_i(\mathcal{M}_i(f_1, \dots, f_{i-1}, f'_i, f_{i+1}, \dots, f_n)) < v_i(\mathcal{M}_i(f_1, \dots, f_{i-1}, f_i, f_{i+1}, \dots, f_n)).$$

Notice that the adverb “weakly” describes the truthfulness, not the level of risk-averseness of the agents. A weakly risk-averse truthful mechanism deals with agents that are *strongly* risk-averse: an agent will not misreport if there is a chance (s)he will receive a value less than what (s)he will receive when truth-telling. A risk-averse truthful mechanism deals with agents that are *weakly* risk-averse: an agent can afford to receive a less value than what (s)he will receive when truth-telling, as long as this value is not below the proportional value. Naturally, a mechanism that is incentive-compatible to agents with weaker risk-averseness achieves a stronger truthful guarantee.

We remark that there is a common Bayesian model capturing the uncertainty of other agents' private information: define a probability distribution from which an agent believes that the other agents' private information is drawn (typically, this distribution depends on the information this agent has). However, in our case, we do not see any natural way to define a probability distribution over piecewise constant functions.

## 5 Risk-Averse Truthful Envy-Free Mechanisms

There exists a simple algorithm that outputs envy-free allocations for  $n$  agents with piecewise-constant value density functions. The algorithm first collects all the points of discontinuity from all agents. This partition the cake into multiple intervals where each agent's value density function is uniform on each of these intervals. Then, the algorithm uniformly allocates each interval to all agents. The output allocation  $(A_1, \dots, A_n)$  of this algorithm satisfies  $v_i(A_j) = \frac{1}{n}v_i([0, 1])$  (this property of an allocation is called *exact*), which is clearly envy-free. However, to make the algorithm deterministic, we need to specify a left-to-right order of the  $n$  agents on how each interval is allocated. The algorithm is described in Algorithm 1.

However, this algorithm is not even weakly risk-averse truthful.

**Theorem 5.1.** *Algorithm 1 is not weakly risk-averse truthful.*

*Proof.* Consider  $f_1$  such that  $f_1(x) = 1$  for  $x \in [0, \frac{1}{n})$  and  $f_1(x) = 0.5$  for  $x \in [\frac{1}{n}, 1]$ , and consider  $f'_1(x) = 1$  for  $x \in [0, 1]$ . Let  $\mathcal{M}$  be the mechanism described by Algorithm 1. We aim to show that, 1) there exist  $f_2, \dots, f_n$  such that  $v_1(\mathcal{M}_1(f'_1, f_2, \dots, f_n)) > v_1(\mathcal{M}_1(f_1, f_2, \dots, f_n))$ , and 2) for any  $f_2, \dots, f_n$ ,  $v_1(\mathcal{M}_1(f'_1, f_2, \dots, f_n)) \geq v_1(\mathcal{M}_1(f_1, f_2, \dots, f_n))$ . That is, misreporting  $f_1$  to  $f'_1$  is sometimes more beneficial and always no harm.

To show 1), consider  $f_2(x) = \dots = f_n(x) = 1$  for  $x \in [0, 1]$ . If agent 1 truthfully reports  $f_1$ , (s)he will receive  $[0, \frac{1}{n^2}) \cup [\frac{1}{n}, \frac{1}{n} + \frac{n-1}{n^2})$ , which is worth  $\frac{1}{n^2} + \frac{n-1}{2n^2}$ . If agent 1 reports  $f'_1$ , the algorithm will see  $n$  uniform functions, and allocation  $[0, \frac{1}{n})$  to agent 1, which is worth  $\frac{1}{n}$ , which is more than  $\frac{1}{n^2} + \frac{n-1}{2n^2}$ .

---

**Algorithm 1** A simple envy-free cake cutting algorithm

---

- 1: let  $X_i$  be the set of all points of discontinuity for  $f_i$
  - 2: let  $X = \bigcup_{i=1}^n X_i$
  - 3: let  $X = \{x_1, \dots, x_{m-1}\}$  be sorted by ascending order, and let  $x_0 = 0, x_m = 1$
  - 4: initialize  $A_i = \emptyset$  for each  $i = 1, \dots, n$
  - 5: **for** each  $j = 0, 1, \dots, m - 1$ :
  - 6:     **for** each agent  $i = 1, \dots, n$ :  $A_i \leftarrow A_i \cup [x_j + \frac{i-1}{n}(x_{j+1} - x_j), x_j + \frac{i}{n}(x_{j+1} - x_j)]$ ;
  - 7: **endfor**
  - 8: **return** allocation  $(A_1, \dots, A_n)$
- 

To show 2), consider any  $f_2, \dots, f_n$ . Suppose agent 1 reports  $f'_1$ . Let  $X$  be defined in Step 2 and 3 of the algorithm with respect to  $f'_1, f_2, \dots, f_n$ . Agent 1 always receives the leftmost  $1/n$  fraction of each  $[x_j, x_{j+1})$ . Since  $f_1$  is monotonically decreasing, this is worth at least  $1/n$  of  $v([x_j, x_{j+1}))$ , and agent 1 receives at least his/her proportional share overall. On the other hand, if agent 1 truthfully reports  $f_1$ , (s)he will always receive exactly his/her proportional share, which is weakly less than what (s)he would receive by reporting  $f'_1$ .  $\square$

The reason for Algorithm 1 not being weakly risk-averse truthful is that an agent can “delete” a point of continuity to merge two intervals  $[x_j, x_{j+1})$  and  $[x_{j+1}, x_{j+2})$ . This may be more beneficial if his/her value is higher on  $[x_j, x_{j+1})$  (or  $[x_{j+1}, x_{j+2})$ ) and (s)he knows that the algorithm will allocate a piece on the very left (or very right) of  $[x_j, x_{j+2})$ . Therefore, it is the deterministic left-to-right order on each interval that compromises the truthfulness. It is easy to randomize Algorithm 1 such that Algorithm 1 is *truthful in expectation*, meaning that an expected utility optimizing agent’s dominant strategy is truth-telling. To achieve this, we just need to partition each  $[x_j, x_{j+1})$  evenly into  $n$  pieces, and allocate these  $n$  pieces to the  $n$  agents by a random matching. This is essentially the algorithm proposed by Mossel and Tamuz [20].

We propose a deterministic risk-averse truthful envy-free mechanism that uses similar ideas. The mechanism is the same as Algorithm 1, except that the left-to-right order on each interval  $[x_j, x_{j+1})$  depends on the index  $j$ . Intuitively, if an agent tries to merges two intervals, (s)he do not know where exactly his/her  $1/n$  fraction of  $[x_j, x_{j+1})$  is, as (s)he does not know other agents’ value density functions. This makes it possible that (s)he ends up receiving a portion where (s)he has less value on. The mechanism is shown in Algorithm 2.

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**Algorithm 2** A risk-averse truthful envy-free cake cutting algorithm

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- 1: let  $X_i$  be the set of all points of discontinuity for  $f_i$
  - 2: let  $X = \bigcup_{i=1}^n X_i$
  - 3: let  $X = \{x_1, \dots, x_{m-1}\}$  be sorted by ascending order, and let  $x_0 = 0, x_m = 1$
  - 4: initialize  $A_i = \emptyset$  for each  $i = 1, \dots, n$
  - 5: **for** each  $j = 0, 1, \dots, m - 1$ :
  - 6:     **for** each agent  $i$ :  $A_i \leftarrow A_i \cup [x_j + \frac{i+j-1 \bmod n}{n}(x_{j+1} - x_j), x_j + \frac{(i+j-1 \bmod n)+1}{n}(x_{j+1} - x_j)]$ ;
  - 7: **endfor**
  - 8: **return** allocation  $(A_1, \dots, A_n)$
- 

**Theorem 5.2.** *Algorithm 2 is risk-averse truthful and envy-free.*

*Proof.* The envy-freeness is trivial. We will focus on risk-averse truthfulness.

We focus on agent 1 without loss of generality. Let  $f_1$  be agent 1’s true value density function. Consider an arbitrarily  $f'_1$  that agent 1 reports. Let  $X_1$  and  $X'_1$  be the sets of all points of discontinuity for  $f_1$  and  $f'_1$  respectively.

Suppose  $X_1 \subseteq X'_1$ . It is easy to see that agent 1 will still get a value of  $\frac{1}{n}v_1([0, 1])$  by reporting  $f'_1$ . This is because any subdivision of an interval where agent 1 has an uniform value gives only smaller intervals each of which agent 1 has an uniform value on. This kind of misreportings is captured by 1 of Definition 4.2.

Suppose  $X_1 \not\subseteq X'_1$ . Pick an arbitrary  $t \in X_1 \setminus X'_1$ . Assume without loss of generality that  $\lim_{x \rightarrow t^-} f(x) < \lim_{x \rightarrow t^+} f(x)$ . Consider a sufficiently small  $\varepsilon > 0$  such that  $[t - \varepsilon, t + (n - 1)\varepsilon]$  do not contain any points in  $X_1 \cup X'_1 \setminus \{t\}$ . We can construct  $f_2, \dots, f_n$  such that 1)  $\bigcup_{i=2}^n X_i$  contains  $X_1 \cup X'_1 \cup \{t - \varepsilon, t + (n - 1)\varepsilon\} \setminus \{t\}$ , 2)  $\bigcup_{i=2}^n X_i$  do not intersect the open interval  $(t - \varepsilon, t + (n - 1)\varepsilon)$ , and 3)  $t - \varepsilon$  is the  $j$ -th point from left to right with  $j$  being a multiple of  $n$ . By our algorithm, agent 1 will receive  $[t - \varepsilon, t)$  on the  $j$ -th interval  $[t - \varepsilon, t + (n - 1)\varepsilon)$ , which is worth less than  $\frac{1}{n}v_1([t - \varepsilon, t + (n - 1)\varepsilon])$ . Agent 1 will receive value exactly  $\frac{1}{n}v_1([0, 1] \setminus [t - \varepsilon, t + (n - 1)\varepsilon])$  on the remaining part of the cake. Therefore, the overall value agent 1 receives is below the proportional value. We have shown that this type of misreportings may cause agent 1's received value less than the proportional value, which corresponds to 2 of Definition 4.2.  $\square$

## 6 Risk-Averse Truthful Proportional Mechanisms with Connected Pieces

Since an entire envy-free allocation is always proportional and Algorithm 2 is entire, Algorithm 2 is risk-averse truthful and proportional. In this section, we are looking for risk-averse truthful proportional mechanisms that satisfy the *connected pieces* property. That is, we require that each agent must receive a connected interval of the cake. Notice that this property is desirable in many applications, e.g., dividing a land.

There are many existing algorithms that output proportional allocations with connected pieces. Two notable algorithms are *the moving-knife procedure* [13] and *Even-Paz algorithm* [15]. We will see in this section that both algorithms are not risk-averse truthful. In particular, the moving-knife procedure is not even weakly risk-averse truthful. We conclude this section by proposing a risk-averse truthful proportional mechanism with connected pieces.

**Moving-knife procedure** Let  $a_i = \frac{1}{n}v_i([0, 1])$  be agent  $i$ 's proportional value. The moving-knife procedure marks for each agent  $i$  a point  $x_i$  such that  $[0, x_i]$  is worth exactly  $a_i$  to agent  $i$ . Then, the algorithm finds the smallest value  $x_{i^*}$  among  $x_1, \dots, x_n$ , and allocates  $[0, x_{i^*}]$  to agent  $i^*$ . Next, for the remaining part of the cake  $[x_{i^*}, 1]$ , the algorithm marks for each of the  $n - 1$  remaining agents a point  $x'_i$  such that  $[x_{i^*}, x'_i]$  is worth exactly  $a_i$  to agent  $i$ . The algorithm then finds the smallest value  $x_{i^\dagger}$  among those  $n - 1$   $x'_i$ s, and allocates  $[x_{i^*}, x_{i^\dagger}]$  to agent  $i^\dagger$ . This is repeated until the  $(n - 1)$ -th agent is allocated an interval, and then the last agent get the remaining part of the cake. It is easy to verify that each of the first  $n - 1$  agents receives an interval that is worth exactly his/her proportional value  $a_i$ , while the last agent may receive more than his/her proportional value.

**Even-Paz algorithm** Even-Paz algorithm is a divide-and-conquer-based algorithm. For each agent  $i$ , Even-Paz algorithm finds a point  $x_i$  such that  $v_i([0, x_i]) = \lfloor \frac{n}{2} \rfloor v_i([0, 1])$ . It then find the median  $x^*$  for  $x_1, \dots, x_n$ . Let  $L$  be the set of agents  $i$  with  $x_i < x^*$  and  $R$  be the set of agents  $i$  with  $x_i \geq x^*$ . Since each agent  $i$  in  $L$  believes  $v_i([0, x^*]) \geq \lfloor \frac{n}{2} \rfloor v_i([0, 1])$  and there are  $\lfloor \frac{n}{2} \rfloor$  agents in  $L$ , there exists an allocation of  $[0, x^*]$  to agents in  $L$  such that each agent  $i$  receives at least his/her proportional value  $\frac{1}{n}v_i([0, 1])$ . For the similar reasons, there exists an allocation of  $(x^*, 1]$  to agents in  $R$  such that each agent  $i$  receives at least his/her proportional value  $\frac{1}{n}v_i([0, 1])$ . The algorithm then solves these two problems recursively. It is also easy to prove that Even-Paz algorithm always outputs proportional allocations.

To show that both algorithms are not risk-averse truthful. We first define the following two value density functions.

$$\ell^{(n)}(x) = \begin{cases} \frac{3}{2} & x \in [0, \frac{1}{2n}) \\ \frac{1}{2} & x \in [\frac{1}{2n}, \frac{1}{n}) \\ 1 & x \in [\frac{1}{n}, 1] \end{cases} \quad r^{(n)}(x) = \begin{cases} 1 & x \in [0, 1 - \frac{1}{n}) \\ \frac{1}{2} & x \in [1 - \frac{1}{n}, 1 - \frac{1}{2n}) \\ \frac{3}{2} & x \in [1 - \frac{1}{2n}, 1] \end{cases} \quad (2)$$

Notice that  $\int_0^1 \ell^{(n)}(x)dx = \int_0^1 r^{(n)}(x)dx = 1$ . The following lemma shows that any allocation that is proportional in either  $\ell^{(n)}$  or  $r^{(n)}$  is also proportional in the uniform value density function.

**Lemma 6.1.** *Let  $f(x) = 1$  for  $x \in [0, 1]$ . For any interval  $I$  such that  $\int_I \ell^{(n)}(x)dx \geq \frac{1}{n}$ , we have  $\int_I f(x)dx \geq \frac{1}{n}$ . For any interval  $I$  such that  $\int_I r^{(n)}(x)dx \geq \frac{1}{n}$ , we have  $\int_I f(x)dx \geq \frac{1}{n}$ .*

*Proof.* We only prove the lemma for  $\int_I \ell^{(n)}(x)dx \geq \frac{1}{n}$ , as the part for  $\int_I r^{(n)}(x)dx \geq \frac{1}{n}$  is similar. It is straightforward to see that  $\int_I \ell^{(n)}(x)dx = \frac{1}{n}$  implies  $|I| \geq \frac{1}{n}$ . In particular,  $|I| = \frac{1}{n}$  if the left endpoint of  $I$  belongs to  $\{0\} \cup [\frac{1}{n}, 1 - \frac{1}{n}]$ , and  $|I| > \frac{1}{n}$  if the left endpoint of  $I$  belongs to  $(0, \frac{1}{n})$ . For  $|I| \geq \frac{1}{n}$ , we have  $\int_I f(x)dx \geq \frac{1}{n}$ . If  $\int_I \ell^{(n)}(x)dx > \frac{1}{n}$ , there exists  $I' \subseteq I$  such that  $\int_{I'} \ell^{(n)}(x)dx = \frac{1}{n}$ . By our previous analysis,  $|I'| \geq \frac{1}{n}$ . We have  $\int_I f(x)dx \geq \int_{I'} f(x)dx \geq \frac{1}{n}$ .  $\square$

**Theorem 6.2.** *The moving-knife procedure is not weakly risk-averse truthful.*

*Proof.* Let  $f_1(x) = 1$  for  $x \in [0, 1]$  be the true value density function for agent 1. We show that agent 1 can misreport his/her value density function to  $f'_1 = \ell^{(n)}$  that satisfies 1) there exists  $f_2, \dots, f_n$  such that  $v_1(\mathcal{M}_1(f'_1, f_2, \dots, f_n)) > v_1(\mathcal{M}_1(f_1, f_2, \dots, f_n))$ , and 2) for any  $f_2, \dots, f_n$ ,  $v_1(\mathcal{M}_1(f'_1, f_2, \dots, f_n)) \geq v_1(\mathcal{M}_1(f_1, f_2, \dots, f_n))$ .

To see 1), suppose  $f_2(x) = 1$  for  $x \in [0, \frac{1}{n}]$  and  $f_2(x) = 0$  for  $x \in (\frac{1}{n}, 1]$ , and  $f_3(x) = \dots = f_n(x) = 0$  for  $x \in [0, \frac{1}{n})$  and  $f_3(x) = \dots = f_n(x) = 1$  for  $x \in [\frac{1}{n}, 1]$ . In the moving-knife procedure, if agent 1 truthfully reports  $f_1$ , (s)he will be the second agent receiving an interval after agent 2 taking  $[0, \frac{1}{n^2})$ , and (s)he will receive  $[\frac{1}{n^2}, \frac{1}{n} + \frac{1}{n^2})$ , which is worth  $\frac{1}{n}$ . If agent 1 reports  $f'_1$ , (s)he will also be the second agent receiving an interval after agent 2 taking  $[0, \frac{1}{n^2})$ , and (s)he will receive  $[\frac{1}{n^2}, \frac{1}{n} + \frac{3}{2n^2})$  (by some simple calculations), which is worth more than  $\frac{1}{n}$  with respect to his/her true valuation.

To see 2), suppose agent 1 reports  $f'_1$ . Since the moving-knife procedure is proportional, regardless of what the remaining  $n - 1$  agents report, agent 1 will receive an interval that has value at least  $\frac{1}{n}$  with respect to  $f'_1$ . By Lemma 6.1, agent 1 receives an interval that is worth at least  $\frac{1}{n}$  with respect to his/her true valuation  $f_1$ . This already shows that the moving-knife procedure is not risk-averse truthful.

We can further show that the procedure is not even weakly risk-averse truthful. Consider any  $f_2, \dots, f_n$ . If agent 1 is not the last agent receiving an interval by reporting  $f_1$  truthfully, agent 1 receives exactly value  $\frac{1}{n}$  by the nature of the moving-knife procedure. Since we have shown that reporting  $f'_1$  also guarantees the proportionality of agent 1, reporting  $f'_1$  will not harm agent 1. Suppose agent 1 is the last agent receiving an interval by reporting  $f_1$  truthfully. Now, suppose agent 1 reports  $f'_1$ . In each iteration of the procedure, by Lemma 6.1, agent 1's marked point for reporting  $f'_1$  is the same as, or on the right-hand side of, agent 1's marked point for reporting  $f_1$ . This indicates that agent 1 will still be the last agent to receive an interval when reporting  $f'_1$ . Moreover, the first  $n - 1$  points cut by the procedure will only depend on  $f_2, \dots, f_n$ . Thus, when agent 1 reports  $f'_1$ , agent 1 receives the same interval as it is in the case where agent 1 reports  $f_1$ . In this case, reporting  $f'_1$  does not harm agent 1 as well.  $\square$

**Theorem 6.3.** *Even-Paz algorithm is not risk-averse truthful.*

*Proof.* Consider the scenario with  $n = 5$  agents. Let  $f_1(x) = 1$  for  $x \in [0, 1]$  be the true value density function for agent 1. We show that agent 1 can misreport his/her value density function to  $f'_1 = r^{(5)}$  that satisfies 1) there exist  $f_2, f_3, f_4, f_5$  such that  $v_1(\mathcal{M}_1(f'_1, f_2, f_3, f_4, f_5)) > v_1(\mathcal{M}_1(f_1, f_2, f_3, f_4, f_5))$ , and 2) for any  $f_2, f_3, f_4, f_5$ , we have  $v_1(\mathcal{M}_1(f'_1, f_2, f_3, f_4, f_5)) \geq \frac{1}{5}v_1([0, 1])$ . Since Even-Paz algorithm is proportional, Lemma 6.1 immediately implies 2). It remains to show 1).

Let  $\varepsilon > 0$  be a small number less than  $\frac{1}{10}$ . Consider  $f_2(x) = 1$  on  $[0, \varepsilon)$  and  $f_2(x) = 0$  on  $[\varepsilon, 1]$ , and  $f_3(x) = f_4(x) = f_5(x) = 0$  on  $[0, 1 - \varepsilon)$  and  $f_3(x) = f_4(x) = f_5(x) = 1$  on  $[1 - \varepsilon, 1]$ . We analyze two cases: the case where agent 1 truthfully reports  $f_1$  and the case where agent 1 reports  $f'_1$ . It is easy to verify that, in both cases, after the first round of the algorithm, an allocation of  $[0, 1 - \frac{3}{5}\varepsilon]$  to agent 1 and 2 is to be decided, and an allocation of  $(1 - \frac{3}{5}\varepsilon, 1]$  to agent 3, 4, 5 is to be decided. In the next round, the algorithm will find the half-half point for each of agent 1 and 2 on  $[0, 1 - \frac{3}{5}\varepsilon]$ , and the algorithm will cut at the median of the two points, which is the average of the two points, and allocate the right-hand side interval to agent 1. By some simple calculations, the half-half point of  $f_1$  on  $[0, 1 - \frac{3}{5}\varepsilon]$  is to the right of the half-half point of  $f'_1$  on  $[0, 1 - \frac{3}{5}\varepsilon]$ . As a result, agent 1 will receive a larger length of interval if (s)he reports  $f'_1$ . Since the true value density function  $f_1$  is uniform, reporting  $f'_1$  will give agent 1 more utility.  $\square$

To conclude this section, we present a mechanism/algorithm that is risk-averse truthful and proportional. In particular, if we require the entire allocations, it is risk-averse truthful for hungry agents. The algorithm is shown in Algorithm 3. Later, we will show that we can modify the algorithm by a little bit to make it risk-averse truthful (without assuming the agents are hungry) if we do not require entire allocations (while still guaranteeing proportionality and connected pieces).

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**Algorithm 3** A risk-averse truthful proportional cake cutting algorithm with connected pieces
 

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- 1: for each  $f_i$ , find  $x_1^{(i)}, \dots, x_{n-1}^{(i)}$  such that  $\int_{x_j}^{x_{j+1}} f_i(x)dx = \frac{1}{n} \int_0^1 f_i(x)dx$  for each  $j = 0, 1, \dots, n-1$ , where  $x_0^{(i)} = 0$  and  $x_n^{(i)} = 1$
  - 2:  $c_0 \leftarrow 0$
  - 3: Unallocated  $\leftarrow \{1, \dots, n\}$  // the set of agents who have not been allocated
  - 4: **for** each  $j = 1, \dots, n-1$ :
  - 5:      $i_j \leftarrow \arg \min_{i \in \text{Unallocated}} \{x_j^{(i)}\}$
  - 6:      $c_j \leftarrow x_j^{(i_j)}$
  - 7:     allocate  $[c_{j-1}, c_j)$  to agent  $i_j$
  - 8:     Unallocated  $\leftarrow \text{Unallocated} \setminus \{i_j\}$
  - 9: **endfor**
  - 10: allocate the remaining unallocated interval to the one remaining agent in Unallocated.
- 

**Theorem 6.4.** *Algorithm 3 is entire and proportional that always outputs allocations with connected pieces.*

*Proof.* It is trivial that the algorithm is entire and always outputs allocations with connected pieces. It remains to show the proportionality. It suffices to show that, in each iteration  $j$ , we have  $[x_{j-1}^{(i_j)}, x_j^{(i_j)}) \subseteq [c_{j-1}, c_j)$  (notice that  $[x_{j-1}^{(i_j)}, x_j^{(i_j)})$  is worth exactly the proportional value for agent  $i_j$ ). Since  $x_j^{(i_j)} = c_j$ , it suffices to show that  $x_{j-1}^{(i_j)} \geq c_{j-1}$ . In the  $(j-1)$ -th iteration, agent  $i_j$  is still in the set Unallocated. Since  $i_{j-1}$  is the agent  $i$  in Unallocated with minimum  $x_{j-1}^{(i)}$ , we have  $x_{j-1}^{(i_j)} \geq x_{j-1}^{(i_{j-1})} = c_{j-1}$ .  $\square$

**Theorem 6.5.** *Algorithm 3 is risk-averse truthful for hungry agents.*

*Proof.* Without loss of generality, we consider the potential misreport for agent 1. Let  $f_1$  be agent 1's true value density function, and consider an arbitrary  $f'_1$ . If the values for  $x_1^{(1)}, \dots, x_{n-1}^{(1)}$  (in Step 1 of the algorithm) are the same for  $f_1$  and  $f'_1$ , the algorithm will output the same allocation for  $f_1$  and  $f'_1$ . In this case reporting  $f'_1$  is not strictly more beneficial. We will conclude the proof by showing that, if the values for  $x_1^{(1)}, \dots, x_{n-1}^{(1)}$  are not the same for  $f_1$  and  $f'_1$ , there exists  $f_2, \dots, f_n$  such that agent 1 will receive an interval with value less than the proportional value (with respect to the true valuation  $f_1$ ).

Suppose  $j^*$  is the minimum index such that  $x_{j^*}^{(1)}$  is not the same for  $f_1$  and  $f'_1$ . Let  $y$  be the value of  $x_{j^*}^{(1)}$  for  $f_1$  and  $y'$  be the value of  $x_{j^*}^{(1)}$  for  $f'_1$ . We consider two cases:  $y' < y$  and  $y' > y$ . Let  $\varepsilon > 0$  be a sufficiently small number.

Suppose  $y' < y$ . We can construct  $f_2, \dots, f_n$  such that 1) for each  $j = 1, \dots, j^* - 1$ ,  $c_j = x_j^{(1)} - \varepsilon$ , and 2)  $c_{j^*} = y'$ . In this case, agent 1 will receive  $[x_{j^*-1}^{(1)} - \varepsilon, y')$ . When  $\varepsilon \rightarrow 0$ , this interval converges to  $[x_{j^*-1}^{(1)}, y')$ , which is a proper subset of  $[x_{j^*-1}^{(1)}, y)$ . We know that  $[x_{j^*-1}^{(1)}, y)$  is just enough to guarantee the proportionality for agent 1. Agent 1 receives an interval with a value less than the proportional value by reporting  $f'_1$ , if  $\varepsilon$  is small enough.

Suppose  $y' > y$ . Since each of the intervals  $[x_0^{(1)}, x_1^{(1)}), \dots, [x_{j^*-2}^{(1)}, x_{j^*-1}^{(1)})$  is worth exactly  $\frac{1}{n}v_1([0, 1])$  and the interval  $[x_{j^*-1}^{(1)}, y')$  is worth strictly more than  $\frac{1}{n}v_1([0, 1])$ , the interval  $[y', 1]$  is worth less than  $\frac{n-j^*}{n}v_1([0, 1])$ . It is possible to find  $y_{j^*+1}, \dots, y_{n-1}$  such that  $[y_j, y_{j+1})$  is worth strictly less than  $\frac{1}{n}v_1([0, 1])$  for each  $j = j^*, \dots, n-1$ , where we let  $y_{j^*} = y'$  and  $y_n = 1$ . Now we construct  $f_2, \dots, f_n$  such that 1)  $c_j = x_j^{(1)} - \varepsilon$  for each  $j = 1, \dots, j^* - 1$ , 2)  $c_{j^*} = y' - \varepsilon$ , and 3)  $\min_i x_j^{(i)} = y_j$  for each  $j = j^* + 1, \dots, n-1$ . It is easy to see that agent 1 will receive an interval that is a subset of one of  $[y_{j^*}, y_{j^*+1}), \dots, [y_{n-1}, 1]$ . Therefore, agent 1 will receive a value less than the proportional value in this case.  $\square$

If the agents are not hungry, it is possible that the set of points  $x_1^{(i)}, \dots, x_{n-1}^{(i)}$  satisfying the condition in Step 1 is not unique. Different selection of this set may result in different allocations. An agent can select this set (by reporting an  $f'_i$  with  $x_1^{(i)}, \dots, x_{n-1}^{(i)}$  being exactly what (s)he want) and potentially receive a better allocation. However, in the case this agent do not know other agents' valuations, it is equal likely

that an agent’s selection is not as good as the algorithm’s default selection. Therefore, *Algorithm 3 is still weakly risk-averse truthful for agents that are not necessarily hungry.*

It is possible to get rid of the hungry agents assumption. The trick is to make sure that each agent  $i$  receives *exactly* one of  $[0, x_1^{(1)}], [x_1^{(1)}, x_2^{(1)}], \dots, [x_{n-1}^{(1)}, 1]$ . In this case, as long as an agent select a set  $x_1^{(i)}, \dots, x_{n-1}^{(i)}$  that satisfies the condition in Step 1, (s)he will get exactly his/her proportional share. Of course, if (s)he select a set  $x_1^{(i)}, \dots, x_{n-1}^{(i)}$  that does not satisfy the condition, the same arguments in the proof of Theorem 6.5 show that there is always a scenario that (s)he will receive a value less than the proportional value. These prove the theorem below, which are stated with the formal proof left to the readers.

**Theorem 6.6.** *If changing Step 7 of Algorithm 3 to “allocate  $[x_{j-1}^{(i_j)}, c_j]$  to agent  $i_j$ ”, Algorithm 3 is risk-averse truthful and proportional (but not entire).*

## 7 Conclusion and Future Work

We have proved that truthful proportional cake cutting mechanism does not exist, even in the restrictive setting with two agents whose value density functions are piecewise-constant and strictly positive. The impossibility result extends to the setting where it is not required that the entire cake needs to be allocated. This resolves the long standing fundamental open problem in the cake cutting literature.

To circumvent this impossibility result and provide a solution that has certain degree of truthfulness in practice, we have proposed a new truthful notion called *risk-averse truthfulness*, which is motivated by the strategy-proofness that the I-cut-you-choose mechanism possesses. We have shown that some well-known cake cutting algorithms do not satisfy this truthful criterion, and we have provided a risk-averse truthful envy-free mechanism, and a risk-averse truthful proportional mechanism with connected pieces.

Below, we discuss a few future directions in this area.

We have proved the impossibility result on truthful proportional mechanisms with  $n = 2$ . Although this implies such mechanisms do not exist in general, it still make senses to consider this problem with a fixed number of agents that is more than 2. The author conjectures that the impossibility result holds for any fixed  $n \geq 2$ .

**Open Problem 1.** Does there exist a positive integer  $n \geq 3$  such that there exists a truthful proportional mechanism with  $n$  agents?

On the other hand, we can relax the proportionality requirement, and instead consider  $\alpha$ -approximation of proportionality.

**Open Problem 2.** Does there exist an  $\alpha > 0$  such that there exists a truthful,  $\alpha$ -approximately proportional mechanism?

Indeed, the author does not even know the existence of a truthful mechanism that guaranteeing each agent a positive value. If the answer to the following open problem is no, we have the same impossibility result as the result of Brânzei and Miltersen [11] for Robertson-Webb query model.

**Open Problem 3.** Does there exist a truthful mechanism that always allocates each agent a subset on which the agent has a positive value?

Of course, if agents are hungry, the answer to the problem above is yes, as the mechanism can just allocate  $[0, 1]$  to the agents such that each agent receives a length of  $\frac{1}{n}$ , disregarding the agents’ reports.

**Empirical Studies** We have proposed two mechanisms that are risk-averse truthful. It is also interesting to test them empirically by simulations or sociological experiments, and compare the performances of them with other classical algorithms such as the moving-knife procedure and Even-Paz algorithm.

## References

- [1] Reza Alijani, Majid Farhadi, Mohammad Ghodsi, Masoud Seddighin, and Ahmad Tajik. Envy-free mechanisms with minimum number of cuts. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 31, 2017.
- [2] Haris Aziz and Simon Mackenzie. A discrete and bounded envy-free cake cutting protocol for four agents. *STOC*, 2016.
- [3] Haris Aziz and Simon Mackenzie. A discrete and bounded envy-free cake cutting protocol for any number of agents. *FOCS*, 2016.
- [4] Haris Aziz and Chun Ye. Cake cutting algorithms for piecewise constant and piecewise uniform valuations. In *International Conference on Web and Internet Economics*, pages 1–14. Springer, 2014.
- [5] Xiaohui Bei, Ning Chen, Xia Hua, Biaoshuai Tao, and Endong Yang. Optimal proportional cake cutting with connected pieces. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 26, 2012.
- [6] Xiaohui Bei, Ning Chen, Guangda Huzhang, Biaoshuai Tao, and Jiajun Wu. Cake cutting: Envy and truth. In *IJCAI*, pages 3625–3631, 2017.
- [7] Xiaohui Bei, Guangda Huzhang, and Warut Suksompong. Truthful fair division without free disposal. *Social Choice and Welfare*, 55(3):523–545, 2020.
- [8] Steven J Brams and Alan D Taylor. An envy-free cake division protocol. *The American Mathematical Monthly*, 102(1):9–18, 1995.
- [9] Steven J Brams, Steven John Brams, and Alan D Taylor. *Fair Division: From cake-cutting to dispute resolution*. Cambridge University Press, 1996.
- [10] Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D Procaccia. *Handbook of computational social choice*. Cambridge University Press, 2016.
- [11] Simina Brânzei and Peter Bro Miltersen. A dictatorship theorem for cake cutting. *IJCAI*, pages 482–488, 2015.
- [12] Y. Chen, J. Lai, D. Parkes, and A. Procaccia. Truth, justice, and cake cutting. In *AAAI*, pages 756–761, 2010.
- [13] Lester E Dubins and Edwin H Spanier. How to cut a cake fairly. *The American Mathematical Monthly*, 68(1P1):1–17, 1961.
- [14] Francis Edward Su. Rental harmony: Sperner’s lemma in fair division. *The American mathematical monthly*, 106(10):930–942, 1999.
- [15] Shimon Even and Azaria Paz. A note on cake cutting. *Discrete Applied Mathematics*, 7(3):285–296, 1984.
- [16] David Kurokawa, John Lai, and Ariel Procaccia. How to cut a cake before the party ends. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 27, 2013.
- [17] Minming Li, Jialin Zhang, and Qiang Zhang. Truthful cake cutting mechanisms with externalities: Do not make them care for others too much! In *Twenty-Fourth International Joint Conference on Artificial Intelligence*, 2015.
- [18] Avishay Maya and Noam Nisan. Incentive compatible two player cake cutting. In *International Workshop on Internet and Network Economics*, pages 170–183. Springer, 2012.
- [19] Vijay Menon and Kate Larson. Deterministic, strategyproof, and fair cake cutting. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence*, pages 352–358, 2017.



- [20] Elchanan Mossel and Omer Tamuz. Truthful fair division. In *International Symposium on Algorithmic Game Theory*, pages 288–299. Springer, 2010.
- [21] A.D. Procaccia. Cake cutting: Not just child’s play. *Communications of the ACM*, 2013.
- [22] Jack Robertson and William Webb. *Cake-cutting algorithms: Be fair if you can*. CRC Press, 1998.
- [23] Walter Stromquist. Envy-free cake divisions cannot be found by finite protocols. *the electronic journal of combinatorics*, 15(1):R11, 2008.
- [24] William Thomson. Problems of fair division and the egalitarian solution. *Journal of Economic Theory*, 31(2):211–226, 1983.